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Decay rate of the Green function in a random potential on the Bethe lattice and a criterion for localization

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Abstract. It is shown that in the tight-binding Anderson model on the Bethe lattice the exponential decay rate of the Green function can be obtained for arbitrary energies and arbitrary disorder. Analytical results in the case of the Lorentzian distribution of site energies are presented. As an application of these results, a criterion for the localized region on the corresponding regular lattice on the level of the Bethe–Peierls approximation is proposed. Our criterion yields exact results in the one-dimensional limit and yields correct band edges for the hypercubic lattice in the vanishing limit of disorder. The mobility edge trajectory obtained by our criterion is given by an elliptic curve in the case of the Lorentzian distribution and its shape is found to be in qualitatively good agreement with that obtained by the finite-size scaling method in the three-dimensional system.

1. Introduction

Since the original work by Anderson [1], much work has been done on the Anderson localization [2, 3]. It is, however, fair to say that clarifying the nature of the localization transition from a statistical mechanical point of view is still a challenging problem. The universality of the critical exponents and the upper critical dimensionality of the localization transition are not yet fully understood.

The localization transition has been studied numerically and analytically by employing the methods which have been developed to treat critical phenomena in statistical mechanics. For example, the numerical finite-size scaling method using quasi-one-dimensional systems has been used to study both the mobility edge trajectory and the critical exponent of the localization length [4–8]. The localization transition in the field-theoretical model, known as the nonlinear σ model, has also been studied using renormalization group theory [9, 10]. The critical exponents for the conductivity and the localization length have been estimated by the $\varepsilon(=d-2)$ expansion of the nonlinear σ model.

On the other hand, mathematical approaches to the Anderson localization have been developed [11–29]. In particular, the one-dimensional case has been studied extensively by mathematical methods [11–16, 19, 26, 29]. Recently, such mathematical methods have also been applied to higher-dimensional systems. The exponential decay of the Green function for sufficiently large disorder or low energies in multi-dimensional systems was first demonstrated by Fröhlich and Spencer [17]. Their work was followed by Martinelli and Scoppola [18] who proved the absence of an absolutely continuous spectrum in the same range of parameters. The condition for the proof of localization has been somewhat relaxed

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by several other authors [19–21]. A new proof for these results has also been given by von Dreifus [22] and Spencer [23]. A rigorous lower bound to the critical exponent for the localization length has been obtained by these mathematical approaches [24, 25]. One of the crucial assumptions in their approach to the proof of localization is the exponential decay of the Green function [19, 26–29]. This has been proved in one dimension for arbitrary energies and arbitrarily small disorder using the positivity of the Lyapunov exponent, but in higher dimensions it has been proved only for sufficiently large disorder or low energies.

The localization transition on the Bethe lattice (the Cayley tree) has also been investigated by several authors [30–35]. The stability of the localized states on the Bethe lattice was first discussed by Abou-Chacra *et al* [30, 31]. Some exact results for the decay of eigenfunctions have been obtained by Kunz and Souillard [32]. It is interesting that these two groups [30–32] obtained the same criterion for the mobility edge, though their approaches seem to be quite different. The Anderson localization on the Bethe lattice was also investigated by Efetov [34, 35] using a field-theoretical model, i.e. the nonlinear σ model.

The Bethe lattice has no closed path and the sites on its boundary are as numerous as those inside the system. Despite these specific properties, it is still interesting to analyse a model on the Bethe lattice since some of its properties may be related to those on a regular lattice, for example, the d-dimensional hypercubic lattice. It is well known that in the case of the Ising model the critical exponents and the critical temperature $T_c^{(BP)}$ obtained by the Bethe–Peierls approximation [36] on the regular lattice are, in fact, reproduced exactly from analysis on the Bethe lattice [37]. From the point of view of statistical mechanics, the Bethe–Peierls approximation as well as the mean-field approximation is important because the critical exponents from these approximations become exact in high dimensions [38, 39]. It has also been shown that the free energy per site in the limit of high dimensionality is exactly that given by the mean-field approximation [40]. The critical exponents in high dimensions are thus also obtained from analysis on the Bethe lattice. This type of relationship between models on the Bethe lattice and those on the regular lattice, in particular in high dimensions, holds for the case of percolation [41], although it appears to be quite different from the Ising model. This has also been discussed in the case of fermionic lattice models [42, 43]. We hence expect that this relationship may also hold for the Anderson transition. However, the mean-field approximation on a regular lattice, which is expected to give exact critical exponents in high dimensions, is not yet known for this transition. The analysis of the Bethe lattice may thus be taken as an appropriate starting point for the study of its mean-field properties including the upper critical dimensionality. As mentioned above, localization on the Bethe lattice has indeed been investigated in this context both by Abou-Chacra *et al* [30, 31] and by Kunz and Souillard [32].

In the present paper, we first show that the exponential decay rate of the Green function at energy $E \in \boldsymbol{R}$ on the Bethe lattice, which is assumed to be sufficiently large but finite, can be obtained by solving a self-consistent nonlinear integral equation for the distribution function of the Green function. It is important to note that we can define the Green function at real energy E, provided that $E \notin \sigma(H)$ where $\sigma(H)$ denotes the energy spectrum of the finite system. The self-consistent equation is obtained from a recursion formula for the Green function. Analytical results in the case of a Lorentzian distribution of site energies are explicitly obtained. As far as we know, except in one dimension, our result for the Bethe lattice is the first rigorous estimation of the exponential decay rate of the Green function for arbitrary energies and arbitrary disorder. In the latter part of the present paper, we propose, as an application of these results, a criterion for the mobility edge on a regular lattice on the level of the Bethe–Peierls approximation, which yields exact results for one-dimensional

systems. Our argument is based on an analogy between the Green function in fermionic systems and the correlation functions in magnetic systems. The mobility edge trajectory obtained by our criterion in the case of a Lorentzian distribution of site energies seems to be in qualitatively good agreement with that obtained by the finite-size scaling method on the corresponding hypercubic lattice. Relationships with the previous work by Abou-Chacra *et al* [30,31] and by Kunz and Souillard [32] are also discussed.

The present paper is organized as follows. In section 2, we introduce a Green function defined by a modified Hamiltonian and derive a self-consistent equation for its distribution function. In section 3, the relation between the decay rate of the Green function and the distribution function of the Green function for a modified Hamiltonian defined in section 2 is explained. Analytical results for a Lorentzian distribution of site energies are explicitly presented in section 4. Our criterion for the mobility edge is described with explicit results for the Lorentzian distribution in section 5. In section 6, the present criterion is discussed in comparison with the criterion from the previous work.

2. Recursion relation for the Green function

We consider non-interacting electrons in a random potential on a Bethe lattice with connectivity K. A Bethe lattice with $K = 2$ is shown in figure 1. On the Bethe lattice, we can define a set of sites $\{j\}$ whose distance from the origin 0 is n (i.e. $|j| \equiv |j - 0| = n$). We call this set of sites the nth shell and it is denoted by S_n in the following. The nth shell consists of $(K+1)K^{n-1}$ sites. The boundary shell is denoted by S_B, where B is the distance between the origin and the boundary and we adopt the free boundary condition. The lattice constant is taken to be one.

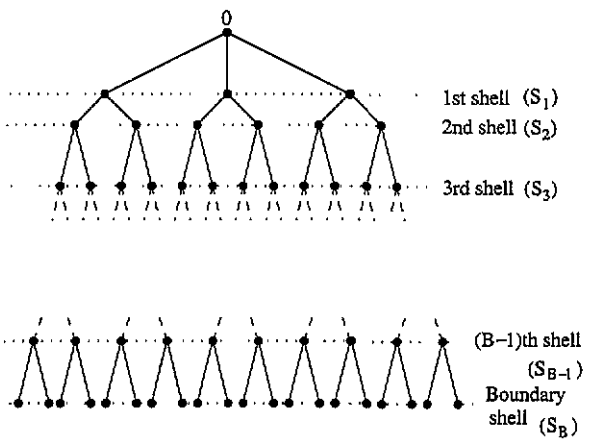

Figure 1. A Bethe lattice with connectivity $K = 2$. The nth shell is denoted by S_n.

We adopt the tight-binding Anderson model defined by the Hamiltonian

$$H = -t \sum_{\langle i,j \rangle} C_i^\dagger C_j + \sum_i V_i C_i^\dagger C_i \tag{2.1}$$

where $C_i^\dagger (C_i)$ is the creation (annihilation) operator of an electron at site i, t denotes the hopping amplitude between nearest neighbours and the site energies are denoted by $\{V_i\}$.

The site energies $\{V_i\}$ are distributed independently and the distribution function is denoted by $P(V)$.

The Green function at energy $E \notin \sigma(H)$ is defined by

$$G(z, z'; E) \equiv \langle z|(E - H)^{-1}|z'\rangle \tag{2.2}$$

where z and z' denote sites on the Bethe lattice and $\sigma(H)$ denotes the energy spectrum of H. Note that the Hamiltonian H is defined on the Bethe lattice whose linear size B is sufficiently large but finite, i.e. $B < \infty$.

Here we introduce an operator $\Gamma_{i,j}$ defined by

$$\Gamma_{i,j} = -t(C_i^\dagger C_j + \text{HC}) \tag{2.3}$$

for the sites i and j with $|i - j| = 1$ and consider a modified Hamiltonian $H_{\Gamma_{i,j}}$ defined by.

$$H_{\Gamma_{i,j}} \equiv H - \Gamma_{i,j}. \tag{2.4}$$

Since there is no hopping amplitude between sites i and j in the Hamiltonian $H_{\Gamma_{i,j}}$, the system described by $H_{\Gamma_{i,j}}$ is divided into two independent parts. The Bethe lattice is accordingly divided into the two corresponding parts: that including site i is denoted by Ω_i and the other including site j by Ω_j (see figure 2).

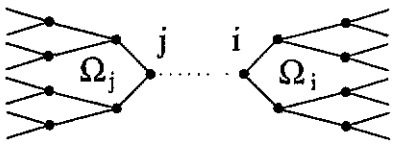

Figure 2. Ω_i and Ω_j of the Bethe lattice in the case $K = 2$, where $i \in \Omega_i$ and $j \in \Omega_j$.

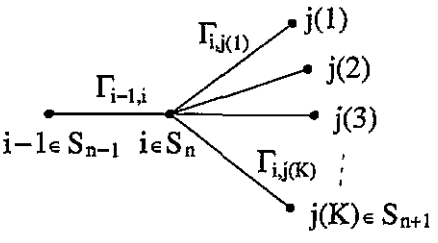

Figure 3. The site $i \in S_n$ and its nearest neighbours $\{(i-1) \in S_{n-1}$ and $j(l) \in S_{n+1}, l = 1, \ldots, K\}$, where K denotes the connectivity of the Bethe lattice.

We then define a Green function $G_{\Gamma_{i,j}}(z, z')$ as

$$G_{\Gamma_{i,j}}(z, z'; E) \equiv \langle z|(E - H_{\Gamma_{i,j}})^{-1}|z'\rangle. \tag{2.5}$$

This Green function $G_{\Gamma_{i,j}}$ is called the modified Green function in the present paper. It is evident from the definition of the Hamiltonian $H_{\Gamma_{i,j}}$ that $G_{\Gamma_{i,j}}(z, z'; E) = 0$ if $z \in \Omega_i$ and $z' \in \Omega_j$.

Let us consider a site $i \in S_n$ and its nearest neighbours $\{i - 1, j(1), \ldots, j(K): i - 1 \in S_{n-1}, j(1), \ldots, j(K) \in S_{n+1}\}$ as shown in figure 3. Using the resolvent identity, we obtain the following recursion relation for the Green function (see appendix A):

$$G_{\Gamma_{i-1,i}}(i, i; E)^{-1} = E - V_i - t^2 \sum_{l=1}^{K} G_{\Gamma_{i,j(l)}}(j(l), j(l); E). \tag{2.6}$$

It should be noted that the modified Green functions $\{G_{\Gamma_{i,j(l)}}(j(l), j(l); E)\}$ on the right-hand side of equation (2.6) are completely independent random variables since the corresponding portions of the Bethe lattice do not share any site. It is obvious that for any site $i \in S_m$ there exists a unique site $j \in S_{m-1}$ satisfying the condition $|i - j| = 1$ and, in this case, the modified Green function $G_{\Gamma_{i,j}}(i, i; E)$ thus, in fact, depends only on site i. This allows us to abbreviate the inverse of the modified Green function to

$$y_i \equiv G_{\Gamma_{i,j}}(i, i; E)^{-1} \tag{2.7}$$

for sites i and j with $|i| > |j|$ and $|i - j| = 1$. The variable y_i is defined for every site except $i = 0$. We can write equation (2.6) in terms of y_i as

$$y_i = E - V_i - t^2 \sum_{l=1}^{K} y_{j(l)}^{-1}. \tag{2.8}$$

It should be remarked here that the quantity y_i satisfying the recursion relation (2.8) is also related to the decay (or growth) rate of the eigenfunctions. The eigenfunction $\psi_E(i)$ is defined as the solution of the Schrödinger equation $H\psi_E = E\psi_E$. We can then easily find that the quantity z_i defined by

$$z_i \equiv -t\psi_E(j)/\psi_E(i) \qquad \text{for } \{i, j : |i| > |j|, |i - j| = 1\} \tag{2.9}$$

also satisfies the same recursion relation as equation (2.8)

$$z_i = E - V_i - t^2 \sum_{l=1}^{K} z_{j(l)}^{-1}. \tag{2.10}$$

Note that in definition (2.9) site j is uniquely determined for fixed i.

Now let us consider the distribution function of the variables $\{y_i\}$, i.e. the inverse of the modified Green functions. If we take the free boundary condition, we have

$$y_m = E - V_m \tag{2.11}$$

for any site m in the boundary shell ($m \in S_B$). The distribution function $Q^{(m)}(y_m)$ of y_m is then given by

$$Q^{(m)}(y_m) = P(E - y_m). \tag{2.12}$$

We thus obtain that the distribution functions of the variables $\{y_m\}$ in the boundary shell ($y_m \in S_B$) are uniform and are equivalent to the distribution function of the site energies. It is easy to see that if the distributions are uniform in the mth shell, the distributions in the $(m-1)$th shell are then also uniform. The distributions are thus uniform in each shell in the case of the free boundary condition. The distribution at the nth shell is denoted by Q_n in the following. We therefore obtain from equation (2.8) the following nonlinear integral equation for the distribution function $Q_n(y_i)$ for $n \neq 0$:

$$\begin{aligned} Q_n(y_i) &= \int P(V_i)\,\mathrm{d}V_i \prod_{l=1}^{K} Q_{n+1}(y_{j(l)})\,\mathrm{d}y_{j(l)}\,\delta\left(y_i - E + V_i + t^2 \sum_{l=1}^{K} y_{j(l)}^{-1}\right) \\ &= \int \prod_{l=1}^{K} Q_{n+1}(y_{j(l)})\,\mathrm{d}y_{j(l)}\,P\left(E - y_i - t^2 \sum_{l=1}^{K} y_{j(l)}^{-1}\right) \\ &\quad i \in S_n,\ j(1), \ldots, j(K) \in S_{n+1}. \end{aligned} \tag{2.13}$$

Next we consider the steady solution $Q(y)$ of equation (2.13). Taking $Q_n = Q_{n+1} \equiv Q$ in equation (2.13), we obtain the following self-consistent nonlinear integral equation

$$Q(y_i) = \int \prod_{l=1}^{K} Q(y_{j(l)})\,\mathrm{d}y_{j(l)}\, P\left(E - y_i - t^2 \sum_{l=1}^{K} y_{j(l)}^{-1}\right) \tag{2.14}$$

for the distribution function $Q(y)$. A nonlinear integral equation of this type has also appeared in previous papers [30, 32] in different contexts. The steady solution $Q(y)$ can be obtained analytically when the distribution function of the random potentials $\{V_i\}$ is Lorentzian. Using this solution $Q(y)$, we discuss the exponential decay rate of the Green function in the next section.

We have discussed the Green function $G_{\Gamma_{i-1,i}}(i, i; E)$ defined for the modified Hamiltonian. In the same way, we can also obtain the distribution Q_0 of the Green function $G(0, 0; E)^{-1}$, where 0 denotes the origin of the Bethe lattice. Since the Green function $G(0, 0; E)$ can be expressed in terms of the modified Green functions $\{G_{\Gamma_{0,j(l)}}\}$ i.e.

$$G(0, 0; E)^{-1} = E - V_0 - t^2 \sum_{l=1}^{Z} G_{\Gamma_{0,j(l)}}(j(l), j(l); E) \qquad j(1), \ldots, j(Z) \in S_1 \tag{2.15}$$

with coordination number $Z \equiv K + 1$, we obtain the following equation

$$Q_0(Y_0) = \int \prod_{l=1}^{Z} Q_1(y_{j(l)})\,\mathrm{d}y_{j(l)}\, P\left(E - Y_0 - t^2 \sum_{l=1}^{Z} y_{j(l)}^{-1}\right) \qquad Y_0 \equiv G(0, 0; E)^{-1} \tag{2.16}$$

for the distribution function $Q_0(Y_0)$ at the origin 0. If the distance from the boundary to the origin is sufficiently large, we may replace Q_1 in equation (2.16) by Q. We then finally arrive at

$$Q_0(Y_0) = \int \prod_{l=1}^{Z} Q(y_{j(l)})\,\mathrm{d}y_{j(l)}\, P\left(E - Y_0 - t^2 \sum_{l=1}^{Z} y_{j(l)}^{-1}\right). \tag{2.17}$$

3. Exponential decay rate of the Green function

Let us consider the Green function $G(m, n; E) \equiv \langle m|(E - H)^{-1}|n\rangle$. On the Bethe lattice, there exists a path from site m to site n which is specified by the sites $\{m, m + 1, \ldots, n = m + L; L = |m - n|\}$ as shown in figure 4. Using the resolvent identity (see appendix A) again, we obtain

$$G(m, m + L; E) = G(m, m + L - 1; E)(-t)G_{\Gamma_{m+L-1,m+L}}(m + L, m + L; E). \tag{3.1}$$

Note that $G_{\Gamma_{m+L-1,m+L}}(m, m + L; E) = 0$, since $\Gamma_{m+L-1,m+L}$ separates the sites m and $m + L$. Repeating this procedure, we obtain

$$G(m, m + L; E) = G(m, m; E) \prod_{j=1}^{L} (-t)G_{\Gamma_{m+j-1,m+j}}(m + j, m + j; E) \tag{3.2}$$

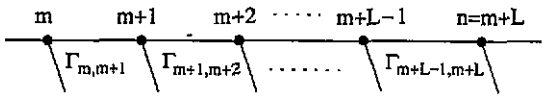

Figure 4. The path from site m to site n ($|n-m|=L$) in the case of $K=2$.

and consequently we have

$$|G(m,m+L;E)| = |G(m,m;E)| \prod_{j=1}^{L} |t||G_{\Gamma_{m+j-1,m+j}}(m+j,m+j;E)|. \tag{3.3}$$

This relation enables us to estimate the exponential decay rate of the Green function $|G(m,m+L;E)|$ using the property of the modified Green functions $\{G_{\Gamma_{m+j-1,m+j}}(m+j,m+j;E)\}$.

For simplicity, we choose m to be the origin 0. Taking the logarithm of equation (3.3), we obtain

$$\frac{1}{L}\ln|G(0,L;E)| = \frac{1}{L}\ln|G(0,0;E)| + \ln|t| + \frac{1}{L}\sum_{j=1}^{L}\ln|G_{\Gamma_{j-1,j}}(j,j;E)|. \tag{3.4}$$

We then assume the following conditions.

Condition 1: $\ln|G(0,0;E)| < \infty$ with probability one.

Condition 2: The distribution function of $\ln|y_i| \equiv \ln|G_{\Gamma_{i-1,i}}(i,i;E)^{-1}|$ has finite moments, i.e. $\langle(\ln|y_i|)^2\rangle < \infty$.

Condition 3: $\langle\ln|y_i|\cdot\ln|y_j|\rangle - \langle\ln|y_i|\rangle\langle\ln|y_j|\rangle) \propto \exp(-a|i-j|)$, for $a > 0$, as $|i-j| \to \infty$.

Here we have used the abbreviation

$$y_i \equiv G_{\Gamma_{j-1,j}}(i,i;E)^{-1}$$

and the triangular bracket $\langle\cdots\rangle$ denotes the arithmetical average over random potentials. The first term of equation (3.4) should go to zero with probability one in the limit $L \to \infty$ under condition 1. Under conditions 2 and 3, we can apply the law of large numbers to the last term of equation (3.4) though the modified Green functions $\{G_{\Gamma_{j-1,j}}(j,j;E)\}$ are not completely independent random variables. We thus obtain

$$\lim_{L\to\infty}\frac{1}{L}\sum_{j=1}^{L}\ln|G_{\Gamma_{j-1,j}}(j,j;E)| = -\langle\ln|y|\rangle \tag{3.5}$$

where $\langle\ln|y|\rangle \equiv \int dy\, Q(y)\ln|y|$ and therefore the quantity on the left-hand side of equation (3.4) is self-averaging; it is independent of the configuration of the random potentials $\{V_i\}$ in the limit as $L \to \infty$. We then finally obtain that the exponential decay rate λ_E of the Green function defined by

$$\lambda_E \equiv -\lim_{L\to\infty}\frac{1}{L}\ln|G(0,L;E)| \tag{3.6}$$

is self-averaging and is given by

$$\lambda_E = -\ln|t| + \langle\ln|y|\rangle. \tag{3.7}$$

It is obvious from equation (3.7) that the decay rate λ_E is identical to the decay rate of the geometrical average of the Green function. The validity of conditions 1, 2 and 3 is essential in the present argument. We verify these three conditions in the case of the Lorentzian distribution in the following section and in appendix B. It is expected that these three conditions will also be valid for many other distributions.

It should be noted here that, in one dimension ($K = 1$), the quantity λ_E is nothing other than the Lyapunov exponent Λ_E [19]. It has already been shown that the exponential decay rate of the eigenfunctions is also given by the Lyapunov exponent Λ_E in one-dimensional systems [15, 16].

If these three conditions are satisfied, the decay rate λ_E is given by equation (3.7) and we immediately obtain

$$\text{Prob}\left(\frac{1}{L}\ln|G(0, L; E)| \geqslant -\lambda_E + \varepsilon\right) \to 0 \qquad \text{as } L \to \infty \tag{3.8}$$

where ε is an arbitrary positive number. Here we denote the probability of event A by Prob(A). In the same way, we obtain

$$\text{Prob}\left(\frac{1}{L}\ln|G(0, L; E)| \leqslant -\lambda_E - \varepsilon\right) \to 0 \qquad \text{as } L \to \infty. \tag{3.9}$$

From these two statements, it follows that

$$|G(0, L; E)| \propto \exp(-\lambda_E L) \qquad \text{as } L \to \infty \tag{3.10}$$

with probability one.

4. Analytical results for the Lorentzian distribution

In this section we consider the case in which the distribution function of the site energies is Lorentzian. The Lorentzian distribution function $P(V)$ of the site energies $\{V_i\}$ is defined by

$$P(V) = \frac{1}{\pi}\frac{\gamma}{V^2 + \gamma^2} \qquad \gamma > 0. \tag{4.1}$$

In this case, if the distribution function Q_{m+1} of the inverse of the modified Green function at the $(m+1)$th shell is Lorentzian, the distribution Q_m at the mth shell is also Lorentzian. We have already shown in section 2 that the distribution function at the boundary shell, Q_B, is given by

$$Q_B(y) = P(E - y) \tag{4.2}$$

since we have adopted the free boundary condition and therefore Q_B is the Lorentzian distribution. The distribution Q_m at any shell then becomes Lorentzian and thus the distribution Q_j at the jth shell ($j = 1, \ldots, B$) can be expressed using parameters A_j and $W_j > 0$ in the form

$$Q_j(y) = \frac{1}{\pi}\frac{W_j}{(y - A_j)^2 + W_j^2}. \tag{4.3}$$

The Fourier transformation of the distribution $Q_j(x)$ is given by

$$Q_j[k] = \exp(-\mathrm{i}A_j k - W_j|k|) \qquad \text{where} \qquad Q_j[k] \equiv \int \exp(-\mathrm{i}kx) Q_j(x)\,\mathrm{d}x. \tag{4.4}$$

Using equation (4.4), we can transform equation (2.13) as

$$\frac{1}{2\pi}\int \exp(\mathrm{i}ky_i)\,\mathrm{d}k \exp(-\mathrm{i}A_n k - W_n|k|)$$
$$= \frac{1}{2\pi}\int \mathrm{d}k \exp\left\{\mathrm{i}k\left(E - y_i - t^2\sum_{l=1}^{K} y_{j(l)}^{-1}\right) - \gamma|k|\right\}\prod_{l=1}^{K} Q_{n+1}(y_{j(l)})\,\mathrm{d}y_{j(l)} \tag{4.5}$$

where $i \in S_n$, $j(1), \ldots, j(K) \in S_{n+1}$. Changing the variable k to $-k$ on the right-hand side and integrating with respect to $\{y_{j(l)}\}$, we get

$$\frac{1}{2\pi}\int \exp(\mathrm{i}ky_i)\,\mathrm{d}k \exp(-\mathrm{i}A_n k - W_n|k|)$$
$$= \frac{1}{2\pi}\int \exp(\mathrm{i}ky_i)\,\mathrm{d}k \exp\left\{-\mathrm{i}k\left(E - Kt^2\frac{A_{n+1}}{A_{n+1}^2 + W_{n+1}^2}\right)\right.$$
$$\left. - \left(\gamma + Kt^2\frac{W_{n+1}}{A_{n+1}^2 + W_{n+1}^2}\right)|k|\right\}. \tag{4.6}$$

We thus obtain the recursion relation for the parameters (A_n, W_n) as

$$A_n = E - Kt^2\frac{A_{n+1}}{A_{n+1}^2 + W_{n+1}^2} \qquad W_n = \gamma + Kt^2\frac{W_{n+1}}{A_{n+1}^2 + W_{n+1}^2} \tag{4.7}$$

or, equivalently,

$$\omega_n = E + \mathrm{i}\gamma - Kt^2\omega_{n+1}^{-1} \tag{4.8}$$

where $\omega_n \equiv A_n + \mathrm{i}W_n$. Taking $\omega_n = \omega_{n+1} \equiv \omega$ in equation (4.8), we obtain the self-consistent equation

$$\omega = E + \mathrm{i}\gamma - Kt^2\omega^{-1} \tag{4.9}$$

for the parameters (A, W) of the steady distribution $Q(y)$, where $\omega = A + \mathrm{i}W$ and

$$Q(y) = \frac{1}{\pi}\frac{W}{(y - A)^2 + W^2}.$$

The physical solution of equation (4.9) is easily obtained as

$$\omega = \frac{1}{2}\left\{E\left(\frac{\sqrt{x} + \gamma}{\sqrt{x}}\right) + \mathrm{i}\left(\gamma + \sqrt{x}\right)\right\} \tag{4.10}$$

where x is the positive root of

$$x^2 + (E^2 - \gamma^2 - 4Kt^2)x - E^2\gamma^2 = 0. \tag{4.11}$$

The exponential decay rate λ_E of the Green function is thus evaluated from equation (3.7) as

$$\begin{aligned}\lambda_E &= \int \ln|y| Q(y)\,dy - \ln|t| \\ &= \ln|\omega| - \ln|t| \\ &= \frac{1}{2}\ln\left(K\frac{\sqrt{x}+\gamma}{\sqrt{x}-\gamma}\right) > 0. \end{aligned} \tag{4.12}$$

In one dimension ($K = 1$), we obtain

$$\lambda_E = \frac{1}{2}\ln\left(\frac{\sqrt{x}+\gamma}{\sqrt{x}-\gamma}\right) \tag{4.13}$$

where x is the positive root of $x^2 + (E^2 - \gamma^2 - 4t^2)x - E^2\gamma^2 = 0$ in this case. This is nothing other than the Lyapunov exponent in the case of the Lorentzian distribution [11]. It should be noted that λ_E is always positive since $K \geqslant 1$.

The distribution Q_0 of the Green function $G(0,0;E)^{-1}$ is also obtained from equation (2.17) as

$$Q_0(Y_0) = \frac{1}{\pi}\frac{W_0}{(Y_0 - A_0)^2 + W_0^2} \tag{4.14}$$

where the parameters (A_0, W_0) are given by

$$A_0 = E - Zt^2\frac{A}{A^2+W^2} \qquad W_0 = \gamma + Zt^2\frac{W}{A^2+W^2}. \tag{4.15}$$

It should be noted that the probability that the variable $Y_0 \equiv G(0,0;E)^{-1}$ takes a particular value $x \in R$ is zero since the distribution $Q_0(Y_0)$ is continuous and has no singularity. This guarantees condition 1 in the previous section. The convergence of series $\{\omega_n\}$ and the validity of condition 3 are discussed in appendix B.

In this case, we can easily show that condition 2 is satisfied. Since we know that the distribution $Q(y)$ of the inverse of the modified Green function is Lorentzian, the distribution function of $x = \ln|y|$ becomes

$$\begin{aligned}F(x) &\equiv \int dy\, Q(y)\delta(x - \ln|y|) \\ &= \frac{W}{\pi}\left(\frac{e^x}{(e^x - A)^2 + W^2} + \frac{e^x}{(e^x + A)^2 + W^2}\right). \end{aligned} \tag{4.19}$$

It is easily seen that

$$F(x) \propto \exp(-|x|) \qquad \text{as } |x| \to \infty$$

and consequently the distribution $F(x)$ has finite moments.

5. A criterion for the localized region

Let us consider here a criterion for the localized region on the regular lattice on the level of the Bethe–Peierls approximation. For this, it may be useful to recall the case of classical spin models or percolation, where the relationship between the critical behaviour of models on the Bethe lattice and its Bethe–Peierls approximation on the regular lattice is well known [37,41,44]. For simplicity, we consider here the ferromagnetic Ising model as an example:

$$H_{\text{Ising}} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \qquad J > 0 \tag{5.1}$$

where $\sigma_i = \pm 1$ denotes the Ising variable. It is well known that the critical temperature in the Bethe–Peierls approximation on the d-dimensional hypercubic lattice can be reproduced from the analysis on the Bethe lattice through the condition that

$$\lambda_{\text{s}}^{\text{Ising}} \equiv \lambda_{\text{T}}^{\text{Ising}} - \ln K = 0 \tag{5.2}$$

where the decay rate $\lambda_{\text{T}}^{\text{Ising}}$ of the correlation function on the Bethe lattice ($K = 2d - 1$) is defined by

$$\lambda_{\text{T}}^{\text{Ising}} \equiv - \lim_{L \to \infty} \frac{1}{L} \ln \langle \sigma_0 \sigma_L \rangle_{\text{T}}. \tag{5.3}$$

Here $\langle\ \rangle_{\text{T}}$ denotes the thermal average. Note that all the quantities which appear in equation (5.2) are defined on the Bethe lattice (the coordination number $K = 2d - 1$). The quantity $\lambda_{\text{s}}^{\text{Ising}}$ introduced in equation (5.2) is the decay rate of the summation of the correlation function over the Lth shell defined by

$$C^{\text{Ising}}(L) \equiv \sum_{j \in S_L} \langle \sigma_0 \sigma_j \rangle_{\text{T}} \tag{5.4}$$

which can be called a surface-to-centre correlation. This surface-to-centre correlation usually plays an important role in various classical spin models and percolation when the correspondence between the Bethe lattice and the regular lattice in such models is discussed [37,41,44].

In the case of the Anderson transition, we may consider that the Green function will correspond to the correlation function in spin systems. We thus expect that in the Anderson transition the summation $C(L)$ of the Green functions over the sites in the Lth shell defined by

$$C(L) \equiv \sum_{j \in S_L} |G(0, j; E)| \tag{5.5}$$

also plays an important role when we consider an approximation for the regular lattice. In the absence of random potentials, for example, the condition that the exponential decay rate of $C(L)$ is equal to zero indeed yields the correct band edges on the d-dimensional hypercubic lattice $E = \pm 2dt$, provided that the connectivity of the Bethe lattice is taken as $K = 2d - 1$. In a random potential, therefore, it is expected that the condition that the decay rate of $C(L)$ is equal to zero will also give a good approximation for the mobility edge on the regular lattice. It is, however, difficult to estimate the decay rate of $C(L)$ itself since it is a summation over a large number of the Green functions. In the arguments developed

in the previous sections, we have shown that each Green function $|G(0, j; E)|$ decays exponentially with a decay rate λ_E. We thus introduce here the quantity $\lambda_s \equiv \lambda_E - \ln K$ as an approximation for the decay rate of $C(L)$. Precisely, λ_s is the decay rate of the summation $C_g(L)$ of the geometrical average of the Green functions defined by

$$C_g(L) \equiv \sum_{j \in S_L} \langle |G(0, j; E)| \rangle_g \tag{5.6}$$

namely

$$C_g(L) \propto \exp(-\lambda_s L) \qquad \text{as } L \to \infty. \tag{5.7}$$

Here $\langle\ \rangle_g$ denotes the geometrical average over random potentials.

On the basis of these considerations, we expect that the condition

$$\lambda_s \equiv \lambda_E - \ln K > 0 \tag{5.8}$$

will yield a reasonable estimate for the localized region on the d-dimensional hypercubic lattice (Z^d) where $d = (K+1)/2$. In our approximation, the mobility edge is given by the condition that $\lambda_s = 0$. It is obvious that this will give exact results in one-dimensional systems. In the following, we present explicitly the analytical result obtained for the Lorentzian distribution.

In the case of the Lorentzian distribution of site energies, we obtain from equation (4.12) that

$$\lambda_s \equiv \lambda_E - \ln K = \frac{1}{2} \ln \left(\frac{1}{K} \frac{\sqrt{x} + \gamma}{\sqrt{x} - \gamma} \right). \tag{5.9}$$

The condition that $\lambda_s > 0$ is thus reduced to

$$\frac{1}{K} \frac{\sqrt{x} + \gamma}{\sqrt{x} - \gamma} > 1 \tag{5.10}$$

where x is the positive root of

$$x^2 + (E^2 - \gamma^2 - 4Kt^2)x - \gamma^2 E^2 = 0. \tag{5.11}$$

It is easy to show that condition (5.10) becomes

$$\frac{E^2}{(K+1)^2} + \frac{\gamma^2}{(K-1)^2} > t^2. \tag{5.12}$$

The region where $\lambda_s > 0$ in the $E - \gamma$ plane is thus obtained as shown in figure 5. The boundary $\lambda_s = 0$ of this region, which is the mobility edge trajectory in our approximation, is given by

$$\frac{E^2}{(K+1)^2} + \frac{\gamma^2}{(K-1)^2} = t^2. \tag{5.13}$$

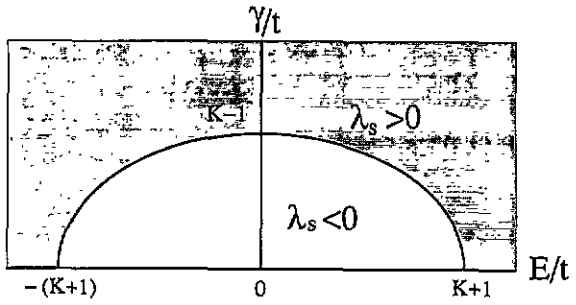

Figure 5. The mobility edge trajectory obtained by our criterion for the Lorentzian distribution $P(V) = \gamma/\{\pi(V^2 + \gamma^2)\}$ in the system with coordination number $Z = K + 1$. Here $\lambda_s = \lambda_E - \ln K$. The region where $\lambda_s > 0$ is represented by the shaded area.

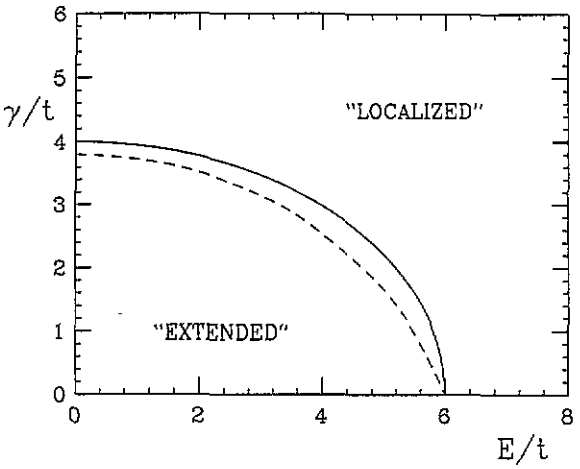

Figure 6. The mobility edge trajectory obtained by our criterion for the Lorentzian distribution $P(V) = \gamma/\{\pi(V^2+\gamma^2)\}$ at $Z = K+1 = 6$ (full curve). The broken curve is the corresponding result obtained by the finite-size scaling method which is taken from the paper by Bulka *et al* [6] where the critical value γ_c at $E = 0$ was estimated to be $\gamma_c/t = 3.8 \pm 0.5$.

Let us look at this result in detail. First, consider the case $\gamma = 0$, i.e. the regular system. In this case, the mobility edges are given by $E = \pm(K + 1)t = \pm Zt$ and consequently we obtain the correct band edges for the d-dimensional ($2d = Z$) hypercubic lattice (Z^d) [30, 31]. In the one-dimensional case ($K = 1$), we recover the rigorous result that eigenstates are localized for any E provided that γ is positive [11–16, 26–29]. We have no region where $\lambda_s < 0$ in the $E - \gamma$ plane in the case $K = 1$. It should also be noted that the shape of the boundary ($\lambda_s = 0$) is consistent with that of the mobility edge trajectory obtained numerically in the three-dimensional system (Z^3) (figure 6) [5, 6]. It has been considered that the absence of extended states outside the unperturbed band is one of the characteristic features of the mobility edge trajectory for the Lorentzian distribution [6, 45] and, in fact, this is clearly seen in our result (figures 5 and 6). We find from equation (5.13) that the critical value γ_c/t for $E = 0$ and $Z = 6$ is given by $K - 1 = 4$. This is close to the value ~ 3.8 [6, 45] obtained by the finite-size scaling method in the three-dimensional system with coordination number 6.

If we define the localization length $\xi(E)$ by the inverse of the decay rate λ_s, we obtain

$$\xi(E) \equiv \lambda_s^{-1} = \left[\frac{1}{2}\ln\left(\frac{1}{K}\frac{\sqrt{x}+\gamma}{\sqrt{x}-\gamma}\right)\right]^{-1}. \tag{5.14}$$

The critical exponent ν for the localization length is defined by

$$\xi(E) \approx (E - E_c)^{-\nu} \qquad E \sim E_c \tag{5.15}$$

where E_c denotes the mobility edge. We then find that the critical exponent ν is given by $\nu = 1$ in our approximation. This value for the critical exponent has been obtained in previous work on the localization transition in the Bethe lattice [32, 34, 35].

One of the characteristic features of the Bethe lattice is that it has no closed path. The physical meaning of our approximation would then be interpreted as neglecting the effect of interferences between different paths and as taking into account only the effect of transmissions and reflections. Our approximation is therefore expected to work well for large $d(= (K+1)/2)$, where the effect of closed loops becomes less important.

6. Discussion

We have shown that the exponential decay rate of the Green function in a random potential on the Bethe lattice can be obtained by solving the self-consistent nonlinear integral equation (2.14) for the distribution function of the modified Green functions. The decay rate λ_E can be considered to be a generalization of the ordinary Lyapunov exponent Λ_E. The self-consistent equation (2.14) has been derived from the recursion relation for the modified Green functions. We have presented explicit results for the Lorentzian distribution of site energies by solving the nonlinear equation analytically. As an application of these results, we have proposed criterion (5.8) for the localized region on regular lattices, which yields exact results in one dimension, and we have obtained qualitatively reasonable results for the mobility edge trajectory on a regular lattice in the case of a Lorentzian distribution of the site energies.

Our criterion for the localized region on the d-dimensional hypercubic lattice is given by

$$\lambda_s \equiv \lambda_E - \ln K > 0 \tag{6.1}$$

using the decay rate λ_E defined by (3.6) and $K = 2d - 1$. On the other hand, the condition under which Kunz and Souillard [32] proved localization on the Bethe lattice can be expressed as (see appendix C)

$$\lambda_s^a \equiv \lambda_E^a - \ln K > 0 \tag{6.2}$$

with the decay rate of the arithmetic average of the Green function defined by

$$\lambda_E^a \equiv -\lim_{L\to\infty}\frac{1}{L}\ln\langle|G(0, L; E)|\rangle_a \tag{6.3}$$

where $\langle\ \rangle_a$ denotes the arithmetic average over random potentials. Abou-Chacra *et al* [30, 31] also obtained condition (6.2) as the condition for the stability of localized states on the Bethe lattice. In general, the decay rate of the Green function (or eigenfunctions) defined by

$$-\lim_{L\to\infty}\frac{1}{L}\ln|G(0, L; E)| \qquad \left(\text{or} \quad -\lim_{L\to\infty}\frac{1}{L}\ln|\psi_E(0)\psi_E(L)|\right) \tag{6.4}$$

is expected to be larger than that of its arithmetic average. It means that $\lambda_s \geqslant \lambda_s^a$ and thus the condition for the localized region is relaxed in our criterion. Our criterion is therefore expected to yield a better approximation for the mobility edge trajectory in the d-dimensional hypercubic lattice ($d = (K+1)/2$) compared with criterion (6.2) obtained by Abou-Chacra *et al* [30] and by Kunz and Souillard [32] which seems to underestimate the localized region. In fact, in the case of the Lorentzian distribution, we have shown that our criterion is able to reproduce a typical feature of the mobility edge trajectory on the corresponding hypercubic lattice which has been obtained numerically by the finite-size scaling method [6]. We note that criterion (6.2), obtained in previous work [30, 32], failed to reproduce this property of the mobility edge trajectory [31]. In the case of the Anderson transition it is usually considered that the effect of interferences between different paths is important, and hence it is remarkable that we have obtained a qualitatively good estimate of the mobility edge trajectory on the regular lattice from an analysis of the Bethe lattice. Our present results may reflect some properties specific to the Lorentzian distribution and thus further investigation of other distributions of site energies is necessary to check the validity of our approximation. Numerical results for the Gaussian and box distributions will be reported elsewhere [46].

We have confined ourselves to the Green function at real energy in this paper. The Green function is usually defined for energy with an imaginary part and physical quantities can be expressed by such Green functions. It is known, however, that localization at energy $E \in \boldsymbol{R}$ on the hypercubic lattice has been proved by von Dreifus and Klein [27] under the condition that the Green function at real energy E decays exponentially in a sufficiently large but finite system, provided that the distribution function of the site energies is absolutely continuous with a bounded density. On a regular lattice, therefore, the Green function at real energy E in a sufficiently large but finite system is indeed directly related to localization at energy E. Since we have tried to propose a criterion for the localized region on a regular lattice, we have also used Green functions at real energy, although our argument is based on analysis of the Bethe lattice.

We have not discussed localization itself on the Bethe lattice. In fact, it is not clear whether our criterion can be related to the condition for localization on the Bethe lattice with $K \geqslant 2$. On the other hand, Kunz and Souillard [32] proved localization on the Bethe lattice under condition (6.2). However, condition (6.2) might not be the necessary condition for localization on the Bethe lattice. In the one-dimensional case ($K = 1$), condition (6.2) reduces to the condition that $\lambda_E^a > 0$ and localization is proved under this condition by Delyon *et al* [13]. Their estimate of λ_E^a is, however, smaller than the Lyapunov exponent Λ_E. Since localization in one dimension is proved under the condition that $\Lambda_E > 0$ [19, 27, 29], this fact would imply that the condition $\lambda_E^a > 0$ is not necessarily the necessary condition for localization in this case ($K = 1$). Whether or not condition (6.2) is the necessary condition for localization on the Bethe lattice with $K \geqslant 2$ remains an open question.

Acknowledgments

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Appendix A. Derivation of equation (2.6)

The resolvent identity states that

$$\frac{1}{E-H_\Gamma} = \frac{1}{E-H_{\Gamma+\Gamma'}-\Gamma'} = \frac{1}{E-H_{\Gamma+\Gamma'}} + \frac{1}{E-H_{\Gamma+\Gamma'}}\Gamma'\frac{1}{E-H_\Gamma} \tag{A.1}$$

where $H_\Gamma \equiv H-\Gamma$ and $H_{\Gamma+\Gamma'} \equiv H-\Gamma-\Gamma'$. Here let us choose Γ and Γ' as

$$\Gamma = \Gamma_{i-1,i} \qquad \text{and} \qquad \Gamma' = \sum_{l=1}^{K} \Gamma_{i,j(l)} \tag{A.2}$$

(see figure 3). We then obtain the following identity

$$G_{\Gamma_{i-1,i}}(i,i;E) = G_{\Gamma_{i-1,i}+\sum_l \Gamma_{i,j(l)}}(i,i;E) + G_{\Gamma_{i-1,i}+\sum_l \Gamma_{i,j(l)}}(i,i;E)(-t)\sum_{l=1}^{K} G_{\Gamma_{i-1,i}}(j(l),i;E). \tag{A.3}$$

In the same way, we get

$$G_{\Gamma_{i-1,i}}(j(l),i;E) = G_{\Gamma_{i-1,i}+\sum_l \Gamma_{i,j(l)}}(j(l),j(l);E)(-t)G_{\Gamma_{i-1,i}}(i,i;E). \tag{A.4}$$

Note that

$$G_{\Gamma_{i-1,i}+\sum_l \Gamma_{i,j(l)}}(j(l),j(l);E) = G_{\Gamma_{i,j(l)}}(j(l),j(l);E)$$

and

$$G_{\Gamma_{i-1,i}+\sum_l \Gamma_{i,j(l)}}(i,i;E) = (E-V_i)^{-1}.$$

Substituting equation (A.4) into equation (A.3), we obtain

$$G_{\Gamma_{i-1,i}}(i,i;E) = \frac{1}{E-V_i}\left\{1 + t^2\left(\sum_{l=1}^{K} G_{\Gamma_{i,j(l)}}(j(l),j(l);E)\right)G_{\Gamma_{i-1,i}}(i,i;E)\right\} \tag{A.5}$$

and we thus finally arrive at equation (2.6):

$$G_{\Gamma_{i-1,i}}(i,i;E)^{-1} = E - V_i - t^2\sum_{l=1}^{K} G_{\Gamma_{i,j(l)}}(j(l),j(l);E). \tag{A.6}$$

Appendix B. Convergence of the series $\{\omega_n\}$

The recursion relation is given by

$$\omega_n = E + i\gamma - Kt^2\omega_{n+1}^{-1}. \tag{B.1}$$

The above parameter ω_n of the distribution on the boundary $n = B$ is given by (see equation (4.2))

$$\omega_B = E + i\gamma. \tag{B.2}$$

Since γ is positive, we immediately have

$$\operatorname{Im}\omega_m > \gamma > 0 \tag{B.3}$$

for any m. Consequently the right-hand side of the recursion relation is well defined for any n.

Let us introduce the following new variables $\{\alpha_i\}$ defined in

$$\omega_i = t\frac{\alpha_{i-1}}{\alpha_i} \qquad i = 1, \ldots, B. \tag{B.4}$$

We then find that the recursion relation for $\{\alpha_i\}$ becomes

$$\alpha_{i-2} = \frac{E + i\gamma}{t}\alpha_{i-1} - K\alpha_i. \tag{B.5}$$

The initial condition should be chosen as

$$\alpha_B = 1 \qquad \alpha_{B-1} = (E + i\gamma)/t \tag{B.6}$$

so that

$$\omega_B \equiv t\frac{\alpha_{B-1}}{\alpha_B} = E + i\gamma. \tag{B.7}$$

The recursion relation (B.5) can be expressed in the form

$$\begin{pmatrix} \alpha_{i-2} \\ \alpha_{i-1} \end{pmatrix} = T \begin{pmatrix} \alpha_{i-1} \\ \alpha_i \end{pmatrix} \tag{B.8}$$

where

$$T = \begin{pmatrix} (E + i\gamma)/t & -K \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} \alpha_{B-1} \\ \alpha_B \end{pmatrix} = \begin{pmatrix} \omega_B \\ 1 \end{pmatrix}. \tag{B.9}$$

The eigenvalues and eigenvectors of the matrix T are obtained as

$$T \begin{pmatrix} \lambda_\pm \\ 1 \end{pmatrix} = \lambda_\pm \begin{pmatrix} \lambda_\pm \\ 1 \end{pmatrix} \tag{B.10}$$

and

$$\lambda_\pm = \frac{1}{2t}\left(E \pm \frac{\gamma E}{\sqrt{x}} + i(\gamma \pm \sqrt{x})\right) \tag{B.11}$$

where x is the positive solution of

$$x^2 + (E^2 - \gamma^2 - 4Kt^2)x - E^2\gamma^2 = 0. \tag{B.12}$$

Notice that the initial vector is expressed as

$$\begin{pmatrix} \omega_B \\ 1 \end{pmatrix} = c_+ \begin{pmatrix} \lambda_+ \\ 1 \end{pmatrix} + c_- \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix} \qquad c_+ = \frac{\omega_B - \lambda_-}{\lambda_+ - \lambda_-} \qquad c_- = \frac{-(\omega_B - \lambda_+)}{\lambda_+ - \lambda_-}. \tag{B.13}$$

We thus obtain

$$\begin{aligned} \begin{pmatrix} \alpha_{B-L} \\ \alpha_{B-L+1} \end{pmatrix} &= T^{L-1} \begin{pmatrix} \omega_B \\ 1 \end{pmatrix} \\ &= c_+\lambda_+^{L-1} \begin{pmatrix} \lambda_+ \\ 1 \end{pmatrix} + c_-\lambda_-^{L-1} \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix} \end{aligned} \tag{B.14}$$

and therefore

$$\begin{aligned} \omega_{B-L} &= t\frac{\alpha_{B-L}}{\alpha_{B-L+1}} \\ &= t\frac{c_+\lambda_+^L + c_-\lambda_-^L}{c_+\lambda_+^{L-1} + c_-\lambda_-^{L-1}} \\ &= t\frac{c_+\lambda_+ + c_-\lambda_-(\lambda_-/\lambda_+)^{L-1}}{c_+ + c_-(\lambda_-/\lambda_+)^{L-1}}. \end{aligned} \tag{B.15}$$

Since the absolute values of the eigenvalues λ_+ and λ_- are given by

$$|\lambda_+|^2 = K\frac{\sqrt{x} + \gamma}{\sqrt{x} - \gamma} \qquad \text{and} \qquad |\lambda_-|^2 = K\frac{\sqrt{x} - \gamma}{\sqrt{x} + \gamma} \tag{B.16}$$

we have $|\lambda_+| \geqslant |\lambda_-|$ provided that $\gamma > 0$. We thus finally obtain that for $\gamma > 0$

$$\begin{aligned} \lim_{L\to\infty} \omega_{B-L} &= t\lambda_+ \\ &= \frac{1}{2}\left(E + \frac{\gamma E}{\sqrt{x}} + \mathrm{i}(\sqrt{x} + \gamma)\right) \\ &= \omega \end{aligned} \tag{B.17}$$

where $B > L$. Note that ω_{B-L} converges to ω exponentially fast. This means that the distribution function at the shell sufficiently far from the boundary can be regarded as the steady distribution Q.

Next, we consider the correlation between y_i, $i \in S_{m-|i-j|}$ and y_j, $j \in S_m$. We show in the following that the distribution y_i under the restriction that the value of y_j is fixed, which is denoted by $Q(y_i; y_j)$, converges exponentially fast to $Q(y_i)$ as $|i-j| \to \infty$. This leads to condition 3 in section 3. Since the distribution is known to be Lorentzian, the convergence of the distribution is equivalent to the convergence of the parameters (A_i, W_i). We assume that sites i and j are sufficiently far from the boundary so that the distribution of y_k around them is considered to be $Q(y_k)$. In order to fix the variable y_j, we take distribution $Q(y_j)$ to be

$$Q(y_j) = \delta(y_j - x) \qquad x \in R \tag{B.18}$$

i.e. $A_j = x$ and $W_j = 0$. We consider the path from j to i which is specified by the sites $\{j, j+1, \ldots, j+L = i; |i-j| = L\}$ where $j + n \in S_{m-n}$ as shown in figure B1 and the distributions under the condition $y_j = x$ are denoted by

$$Q(y_{j+m}; y_j = x) = \frac{1}{\pi} \frac{\tilde{W}_{j+m}}{(y_{j+m} - \tilde{A}_{j+m})^2 + \tilde{W}_{j+m}^2} \qquad \tilde{\omega}_{j+m} \equiv \tilde{A}_{j+m} + \mathrm{i}\tilde{W}_{j+m}. \tag{B.19}$$

We then get the recursion relation

$$\tilde{\omega}_{j+n+1} = E + \mathrm{i}\gamma - (K-1)t^2\omega^{-1} - t^2\tilde{\omega}_{j+n}^{-1} \tag{B.20}$$

where the initial condition is $\tilde{\omega}_j = x \in \boldsymbol{R}$. Using property (4.9), we get

$$\tilde{\omega}_{j+n+1} = \omega + t^2\omega^{-1} - t^2\tilde{\omega}_{j+n}^{-1}. \tag{B.21}$$

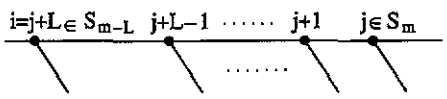

Figure B1. The path from site $j \in S_m$ to site $i \in S_{m-L}$ ($|i-j| = L$) in the case of $K = 2$.

By using almost the same argument as before (see equations (B.4)–(B.17)), we can show that $\tilde{\omega}_{j+L}$ converges to ω exponentially fast:

$$\lim_{L\to\infty} \tilde{\omega}_{j+L} = \omega. \tag{B.22}$$

Appendix C. Relation to previous work

In order to see the relation to previous work [30, 32], let us consider the quantity defined by

$$C(L) \equiv \sum_{j\in S_L} |G(0, j; E)| \tag{C.1}$$

where 0 denotes the origin of the Bethe lattice (see equation (5.5)).

First, we consider the arithmetic average of the quantity $C(L)$

$$\langle C(L)\rangle_{\mathrm{a}} = \sum_{j\in S_L} \langle |G(0, j; E)|\rangle_{\mathrm{a}} \tag{C.2}$$

where the triangular bracket $\langle\cdots\rangle_{\mathrm{a}}$ denotes the arithmetic average with respect to the distribution of the random potentials $\{V_i\}$. Since averaged quantities are uniform in our system, equation (C.2) can be further simplified as

$$\sum_{j\in S_L} \langle |G(0, j; E)|\rangle_{\mathrm{a}} = (K+1)K^{L-1} \langle |G(0, L; E)|\rangle_{\mathrm{a}} \tag{C.3}$$

where the site L on the right-hand site of this equation denotes a site in the Lth shell and the path from the origin 0 to the site L is parametrized by the sites $\{0, 1, 2, \ldots, L\}$ as in section 3. From equation (3.3) we get

$$\langle |G(0, L; E)| \rangle_a = \left\langle |G(0,0)| \prod_{m=1}^{L} |t| |G_{\Gamma_{m-1,m}}(m, m; E)| \right\rangle_a$$
$$= \left\langle |y_0|^{-1} \prod_{m=1}^{L} |t| |y_m|^{-1} \right\rangle_a \tag{C.4}$$

where $m \in S_m$, $y_0 = G(0, 0; E)^{-1}$ and $y_m = G_{\Gamma_{m-1,m}}(m, m; E)^{-1}$. It is then easy to see that the exponential decay rate of the arithmetic average of the Green function is given as

$$-\lim_{L\to\infty} \frac{1}{L} \ln \langle |G(0, L; E)| \rangle_a = -\ln \lambda_{\max} \tag{C.5}$$

where $\lambda_{\max}$ denotes the largest eigenvalue of the transfer operator T defined by

$$T(x; y) \equiv \int \frac{|t|}{|y|} \prod_{i=1}^{K-1} Q(y_i)\, dy_i\, P\left(E - x - t^2 y^{-1} - t^2 \sum_{i=1}^{K-1} y_i^{-1}\right). \tag{C.6}$$

We thus obtain that the condition for the exponential decay of the arithmetic average of the summation of the Green function $\langle C(L) \rangle_a$ is given by $K\lambda_{\max} < 1$.

The properties of operator T in (C.6) and its eigenvalues were discussed by Kunz and Souillard [32]. They discussed the correlation of eigenfunctions instead of the Green function and found that the decay of the arithmetic average of the correlation of eigenfunctions at energy E was determined by the largest eigenvalue of the same transfer operator T above defined [32]. Namely,

$$-\lim_{L\to\infty} \frac{1}{L} \ln \langle |\psi_E(0)\psi_E(L)| \rangle_a = -\ln \lambda_{\max}. \tag{C.7}$$

Here ψ_E is the solution of the Schrödinger equation $H\psi = E\psi$ and the bracket $\langle \cdots \rangle_a$ again denotes the arithmetic average over random potentials. As a result, we find that the decay rate of the arithmetic average of the Green function is identical to that of the correlation of eigenfunctions. The reason why we have obtained the same decay rate for both the Green function and the correlation function is that the ratio of the eigenfunction z_i defined by

$$z_i \equiv -t\psi_E(j)/\psi_E(i) \qquad \text{for } \{i, j : |i| > |j|, |i - j| = 1\}$$

and the modified Green function

$$G_{\Gamma_{j,i}}(i, i; E)^{-1} \qquad \text{for } \{i, j : |i| > |j|, |i - j| = 1\}$$

satisfy the same recursion relation as mentioned in section 2. Note that the correlation of eigenfunctions is also expressed by the product of the variables $\{z_i\}$ as

$$|\psi_E(0)\psi_E(L)| = |\psi_E(0)|^2 \prod_{i=1}^{L} \frac{|t|}{|z_i|} \tag{C.8}$$

(compare with equation (3.3) in section 3). They claimed that they proved localization under the condition $K\lambda_{\max} < 1$, namely the exponential decay of the summation of the correlation functions over the sites in the Lth shell [32]:

$$\langle C_\psi(L)\rangle_a \equiv \sum_{j\in S_L} \langle|\psi_E(0)\psi_E(j)|\rangle_a \propto \exp(-aL) \qquad a > 0 \tag{C.9}$$

in the limit as $L \to \infty$.

As was pointed out by Kunz and Souillard [32], the condition that $K\lambda_{\max} < 1$ is equivalent to the condition for the stability of the localized states obtained by Abou-Chacra *et al* [30]. They considered that the energy E lies in the localized region when the imaginary part of the self-energy vanishes as the imaginary part of the energy goes to zero and they investigated the distribution function of the self-energy. The distribution function of the real part of the self-energy they considered is directly related to the distribution Q we consider in the present paper, but they did not mention the decay of the Green function. They discussed the stability of the desired solution of the distribution function of the self-energy for the localized state. We then find that their condition for the stability of the localized states [30] is understood as the condition for $\langle C(L)\rangle_a \propto \exp(-aL)$, $a > 0$.

The criterion for the mobility edge was thus given by $K\lambda_{\max} = 1$ in the previous work by Abou-Chacra *et al* [30] and Kunz and Souillard [32]. It should be emphasized that this criterion for the mobility edge is based on the property of the arithmetic average of the Green function or eigenfunctions.

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